Nonlinear Membership Functions in Multiobjective Fuzzy Optimization of Mechanical and Structural Systems

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An application of fuzzy mathematical programming techniques to multiple objective design problems is presented. Two examples dealing with the multiobjective design of mechanical and structural systems are considered. The concept of a Pareto-optimal and fuzzy Pareto-optimal solution is discussed, and it is shown that the resulting formulation yields Pareto-optimal solutions. The fundamental assumption in fuzzy mathematical programming applications involving the use of linear membership functions is critically examined. Several nonlinear shapes for the membership functions of the fuzzy sets are proposed, consistent with varying perceptions of the designer, and are analyzed to determine their impact on the overall design process. These shapes correspond to what we define as the coefficient of membership satiation. It is seen that optimum designs for both examples are strongly influenced by the sign of the membership satiation coefficient.

I. Introduction

MODERN computer-aided design methods for engineering systems generally assume that a scalar objective function, or functional such as cost, efficiency, or weight, can be defined, so that standard computational algorithms from mathematical programming or optimal control can be applied to obtain an optimum solution. The usefulness of these methods is seriously limited by the fact that the performance of a complex engineering system depends on a number of different, often conflicting, criteria that cannot be combined into a single measure of performance. Hence, consideration of multiple objective functions becomes an important aspect in the design of engineering systems.

Furthermore, in modeling real world design problems, a designer is often forced to state a problem in precise mathematical terms rather than in real world terms, which are often imprecise in nature. The relationships and statements used for problem description may be imprecise, not due to randomness, but because of inherent fuzziness in the system. With increasing system complexity, one's ability to make precise and significant statements concerning a given system diminishes further.[18] Consequently, the closer one examines a real world problem, the fuzzier its description becomes. Fuzzy set theories can effectively model such domains in which the description of activities and observations are "fuzzy," in the sense that there are no sharply defined boundaries of the set of activities or observations to which the descriptions apply. These theories enable one to structure and describe activities that differ from each other vaguely, to formulate them in models, and to use these models for problem solving and decision making.

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The first application of fuzzy theories to decision making processes was presented by Bellman and Zadeh.[1] This paper prescribed basic concepts and definitions associated with a decision-making process in a fuzzy environment. Since then, these conceptual techniques have been employed to formulate and solve several mathematical programming problems. An application of fuzzy optimization techniques to linear programming problems with single and multiple objectives functions has been presented by Zimmermann.[19,20] Hannan has applied these theories to preemptive and Archimedian versions of goal programming problems.[7] Wang and Wang have used the method of level cut solutions for the fuzzy optimum design of structures.[17] An application of fuzzy optimization tech-

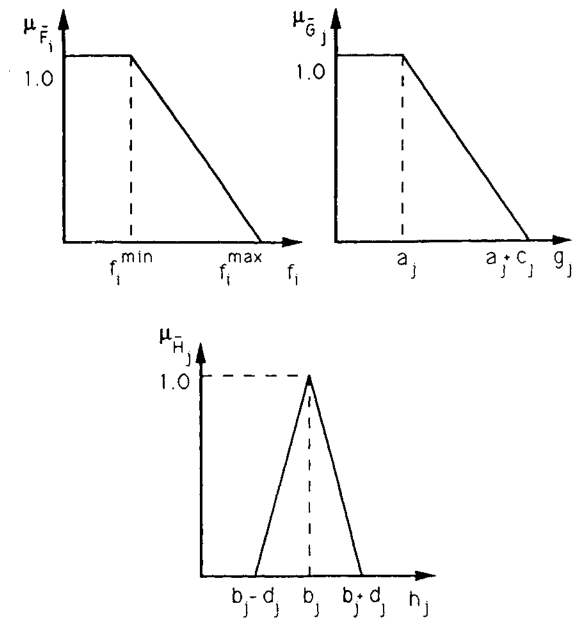

Fig. 1 Linear membership functions: a) fuzzy objective function, b) fuzzy inequality constraint, c) fuzzy equality constraint.

niques to the design of mechanical and structural systems has been discussed by Rao.[14,15] The concept of efficient and weakly efficient solutions in the context of fuzzy vector maximum problems has also been discussed by Feng[5] and Negotia.[12]

The purpose of this research is manifold. Some of the key aspects of the work are as follows: 1) to illustrate the applicability of fuzzy multiobjective mathematical programming techniques to engineering design problems; 2) to present a new multiobjective formulation for the design of high-speed planar mechanisms that integrates both the kinematic and dynamic synthesis aspects of design; and 3) to develop new, improved, and more realistic fuzzy mathematical programming models with regard to the concepts of fuzzy Pareto-optimality, and the use of nonlinear membership functions.

Toward this end, two examples dealing with a multiobjective design of mechanical and structural systems are considered. The first problem deals with the design of planar high-speed mechanisms, whereas the second example addresses the optimum design of space structures. It is shown that the fuzzy min-operator, together with linear as well as nonlinear membership functions, yields Pareto-optimal solutions to the original multicriteria problem. Several nonlinear shapes for the membership functions are considered to determine their influence on the overall design process. It is seen that the optimum designs for both examples are strongly influenced by the nature of the designer's behavior with respect to fuzzy objective functions and constraints.

II. Multiple Objective Mathematical Programming

A general multiple objective nonlinear programming (MONLP) problem is of the following form:

Minimize

$$f(X) = [f_1(X), f_2(X), \ldots, f_k(X)]$$

subject to

$$X \in S = [X \mid X \in R^n, \quad g_i(X) \le a_i, \quad h_j(X) = b_j] \tag{1}$$

where X is an n-dimensional vector of design variables, $f_1(X), \ldots, f_k(X)$ are k distinct objective functions, and S is the set of feasible solutions. An optimum solution, for a single objective problem, is defined as one that minimizes the objective function $f_i(X)$ subject to the constraint set $X \in S$. Attempting to define a vector minimal point as one at which all components of the objective function vector f are simultaneously minimized is not an adequate generalization since such "utopia" points are seldom attainable. The concept of a Pareto-optimal and a weakly Pareto-optimal solution has been found to be quite useful in this context.[8,16]

Definitions and Theorems

Definition 1: A feasible solution $X^* \in S$ is Pareto-optimal if there is no $\bar{X} \in S$, such that $f_i(\bar{X}) \le f_i(X^*)$, $i = 1, \ldots, k$, and $f_{i_0}(\bar{X}) \le f_{i_0}(X^*)$ for least one $i_0 \in [1, \ldots, k]$.

Alternately, a design vector X^* is Pareto-optimal if there exists no feasible vector $\bar{X}$ that would decrease some objective function f_i without causing a simultaneous increase in at least one other objective function f_j, $i \ne j$.

Definition 2: A feasible solution $X^* \in S$ is weakly Pareto-optimal if there is no $\bar{X} \in S$, such that $f_i(\bar{X}) < f_i(X^*)$, $i = 1, \ldots, k$.

If X^* is Pareto-optimal, then X^* is weakly Pareto-optimal. The converse is true only under special circumstances.

Unless a problem is convex, only a locally optimal solution is guaranteed using standard mathematical programming techniques. Thus, the concept of Pareto-optimality needs to be modified to introduce the notion of a locally Pareto-optimal solution for a nonconvex problem, as discussed in Ref. 6.

Definition 3: A solution $X^* \in S$ is said to be locally Pareto-optimal if and only if there exists a $\delta > 0$ such that X^* is Pareto-optimal in $S \cap N(X^*, \delta)$ where $N(X^*, \delta)$ denotes a neighborhood of X^*, i.e., the set $[X \mid X \in S, |X - X^*|_2 < \delta]$.

The set of Pareto-optimal solutions usually consists of an infinite number of points, and additional information is needed to order the set of Pareto-optimal solutions. This makes it possible to bring in additional considerations that are not included in the optimization model, thus making the multiobjective approach a flexible technique for most design problems. Utility function, goal programming, interactive, and fuzzy approaches have been proposed to determine an optimal compromise solution from the set of Pareto-optimal solutions.[8,11,21] The utility and goal programming methods entail an a priori articulation of preference information on the part of the decision maker (DM). Interactive methods, on the other hand, require a progressive articulation of preference information.

If the DM does not exhibit a minimizing behavior, but rather acts as a satisfier, the MONLP can be stated as problem P1:

Find X, such that

$$f_i(X) \le a_i, \qquad i = 1, \ldots, k \qquad X \in S \tag{2}$$

where a_i denotes the aspiration levels for the k objective functions. The satisfaction level of the DM increases or decreases monotonically as the design approaches the aspiration levels, depending on whether they represent the lower or upper bounds of acceptability. If $f_i^{\min}$ and $f_i^{\max}$, respectively, represent the lower and upper bounds of aspirations with respect to f_i, then the DM does not accept a solution for which $f_i \ge f_i^{\max}$ and is completely satisfied whenever $f_i \le f_i^{\min}$. Thus one can express the objective functions f_i by fuzzy sets whose membership functions decrease monotonically from 1 at $f_i^{\min}$ to 0 at $f_i^{\max}$. Problem P1 can be rewritten as follows:

Find X, such that

$$f_i(X) \in \bar{F}_i(X), \qquad i = 1, \ldots, k \qquad X \in S \tag{3}$$

where $\bar{F}_i$ denotes the allowable interval for fuzzy objective function f_i, $\bar{F}_i = [f_i^{\min}, f_i^{\max}]$. The overbar indicates the presence of fuzzy information. If a min-operator is employed for aggregating fuzzy sets, an optimum solution to the fuzzy mathematical programming problem [Eq. (3)] is obtained by[5,21] problem P2:

Find X which

$$\max \lambda$$

subject to

$$\lambda \le \mu_{\bar{F}_i}(X), \qquad i = 1, \ldots, k \qquad X \in S \tag{4}$$

where $\mu_{\bar{F}_i}$ are the membership functions corresponding to fuzzy objectives f_i, denoted by fuzzy sets $\bar{F}_i$.

Theorem 1: If X^* is a local optimal solution to problem P2, then any $\bar{X} \in S$ that strictly dominates X^* is also a local optimal solution to problem P2.

Proof: Since $\bar{X}$ strictly dominates X^*, it follows that

$$f_i(\bar{X}) \le f_i(X^*), \qquad \text{for all } i \in [1, \ldots, k] \tag{5}$$

and

$$f_{i_0}(\bar{X}) < f_{i_0}(X^*), \qquad \text{for at least one } i_0 \in [1, \ldots, k] \tag{6}$$

Since the membership functions are monotonically decreasing functions of f_i, Eqs. (5) and (6) yield

$$\mu_{\bar{F}_i}(\bar{X}) \ge \mu_{\bar{F}_i}(X^*), \qquad \text{for all } i \in [1, \ldots, k] \tag{7}$$

and

$$\mu_{\bar{F}_{i_0}}(\bar{X}) > \mu_{\bar{F}_{i_0}}(X^*), \qquad \text{for at least one } i_0 \in [1, \ldots, k] \tag{8}$$

Thus,

$$\min_i \mu_{\tilde{F}_i}(\bar{X}) \geq \min_i \mu_{\tilde{F}_i}(X^*) \tag{9}$$

or

$$\bar{\lambda} \geq \lambda^* \tag{10}$$

If X^* is a local optimal solution to problem P2, then Eq. (10) is an equality, and consequently, $\bar{X}$ is also a local optimal solution to problem P2.

Theorem 2: If X^* is a unique local optimal solution to problem P2, then X^* is a local Pareto-optimal solution to the MONLP given by Eq. (1).

Proof: Since X^* is a unique local optimal solution to problem P2, there exists no $\bar{X} \in S \cap N(X^*,\delta)$, such that

$$f_i(\bar{X}) \leq f_i(X^*), \qquad \text{for all } i \in [1,\ldots,k]$$

and

$$f_{i_0}(\bar{X}) < f_{i_0}(X^*), \qquad \text{for at least one } i_0 \in [1,\ldots,k]$$

because it would contradict the uniqueness of X^* by Theorem 1. Thus, for each $\bar{X} \in S \cap N(X^*,\delta)$, there exists an i_0, such that

$$f_{i_0}(\bar{X}) > f_{i_0}(X^*)$$

This leads us to conclude that there exists no $\bar{X} \in S \cap N(X^*,\delta)$ for which

$$f_i(\bar{X}) \leq f_i(X^*), \quad \text{for all } i \in [1,\ldots,k]$$

and

$$f_{i_0}(\bar{X}) < f_{i_0}(X^*), \qquad \text{for at least one } i_0 \in [1,\ldots,k]$$

Thus, X^* is a local Pareto-optimal solution to the original MONLP given by Eq. (1).

Computational Procedure

An optimal compromise solution to the fuzzy multiobjective problem P2 is determined by 1) finding the solutions to the individual single objective optimization problems, 2) determining the best and worst values for each of the objective functions, 3) using these values as the boundaries of the fuzzy ranges for the fuzzy objective functions in the corresponding optimization problem, and 4) solving the resulting fuzzy optimization problem.

A linear membership function of a fuzzy objective function, for example, is constructed as[1,3]

$$\mu_{\tilde{F}_i}(X) = \begin{cases} 0, & \text{if } f_i(X) \geq f_i^{\max} \\ \left(\dfrac{-f_i(X)+f_i^{\max}}{f_i^{\max}-f_i^{\min}}\right), & \text{if } f_i^{\min} < f_i(X) < f_i^{\max} \\ 1, & \text{if } f_i(X) \leq f_i^{\min} \end{cases} \tag{11}$$

$$i = 1,\ldots,k$$

where $f_i^{\min} = \min_j f_i(X_j^*)$ and $F_{ij} = f_i^{\max} = \max_j f_i(X_j^*)$, and X_j^* is the optimum design vector of the jth objective function. A linear membership function models a DM's constant marginally increasing (or decreasing) membership value over the parameter range of interest, and is defined by fixing upper and lower levels of design parameter acceptability. The linear membership function, as well as six other common nonlinear shapes modeling various marginally increasing and decreasing membership values, are analyzed to determine their impact on the overall design process. These nonlinear shapes are discussed in detail in Sec. III. A generalization of problem P2, when the design constraints are partly crisp and partly fuzzy, will now be addressed.

When the fuzzy constraints are stated as

$$g_j(X) \leq a_j + c_j, \qquad j = 1,\ldots,p \tag{12}$$

where c_j denotes the distance by which the boundary of the jth constraint is moved, the linear membership function for the jth inequality constraint is constructed as

$$\mu_{\tilde{G}_j}(X) = \begin{cases} 0, & \text{if } g_j(X) \geq a_j + c_j \\ 1-\left(\dfrac{g_j(X)-a_j}{c_j}\right), & \text{if } a_j < g_j(X) < a_j + c_j \\ 1, & \text{if } g_j(X) \leq a_j \end{cases} \tag{13}$$

$$j = 1,\ldots,p$$

For a fuzzy equality constraint stated as

$$h_j(X) \leq b_j \pm d_j, \qquad j = 1,\ldots,q \tag{14}$$

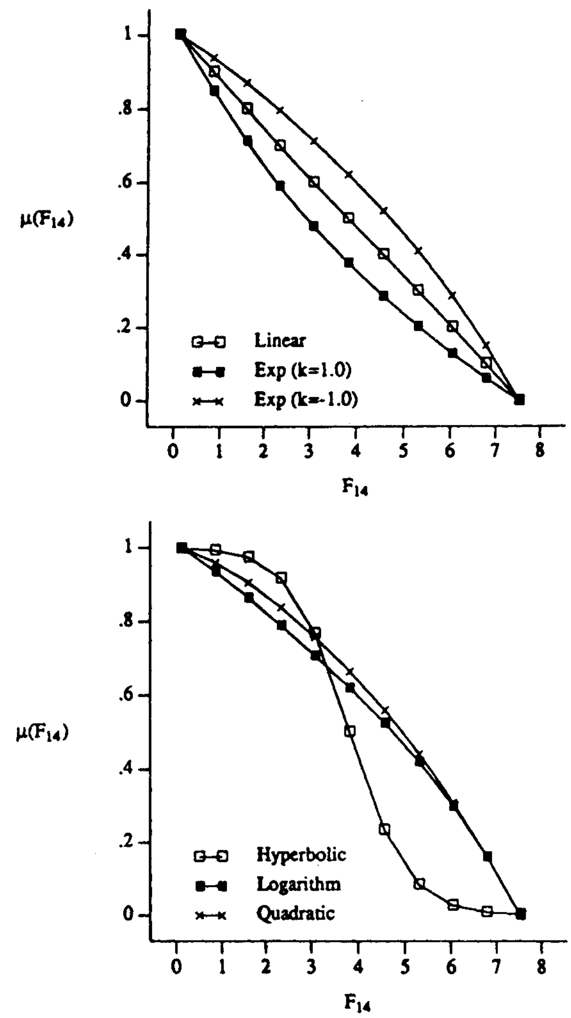

Fig. 2 Nonlinear membership functions.

Table 1 Objective function values for single objective optimizations

Objective function	Starting vector	Min. ϵ[a]	Min. F_{12}[b]	Min. F_{14}[c]	Min. T_s[d]	Min. SF[e]	Min. SM[f]
$\epsilon \times 10^2$	55.242	2.63[g]	65.81	37.64	65.83	17.27	32.4
F_{12}	15.988	15.38	0.0944[g]	22.791	22.198	0.9837	0.1548
F_{14}	8.5648	7.524	0.1469	0.1051[g]	0.4149	0.9663	0.1191
T_s	4.6061	3.654	0.0464	0.1413	0.0401[g]	0.2561	0.0635
SF	18.884	19.34	0.1698	22.882	22.396	0.0561[g]	0.1648
SM	25.479	23.61	0.1489	41.098	35.933	1.5398	0.1235[g]

[a]Minimize structural error.
[b]Minimize maximum value of F_{12}.
[c]Minimize maximum value of F_{14}.
[d]Minimize maximum value of driving torque.
[e]Minimize maximum value of shaking force.
[f]Minimize maximum value of shaking moment.
[g]Optimal single-objective function values.

where d_j denotes the distance by which the boundary of the jth constraint is moved, the linear membership function for the jth equality constraint is constructed as

$$\mu_{\tilde{H}_j}(X) = \begin{cases} 0, & \text{if } h_j(X) \geq b_j + d_j \\ 1 - \left(\dfrac{|h_j(X) - b_j|}{d_j}\right), & \text{if } b_j - d_j < h_j(X) < b_j + d_j \\ 0, & \text{if } h_j(X) \leq b_j - d_j \end{cases} \tag{15}$$

$$j = 1, \ldots, q$$

A linear membership function corresponding to a fuzzy objective function, as well as a fuzzy equality and an inequality constraint, are shown in Fig. 1. Once the membership functions of the fuzzy objective functions and the fuzzy constraints, i.e. $\mu_{\tilde{F}_i}$, $\mu_{\tilde{G}_j}$, and $\mu_{\tilde{H}_j}$ are known, the fuzzy optimization problem P2 can be posed as an equivalent crisp optimization problem as follows:

Problem P3: Find $\bar{X}$ and λ which

$$\max \lambda$$

subject to

$$\begin{aligned} \lambda &\leq \mu_{\tilde{F}_i}(X), & i &= 1, \ldots, k \\ \lambda &\leq \mu_{\tilde{G}_j}(X), & j &= 1, \ldots, p \\ g_j(X) &\leq a_j, & j &= p+1, \ldots, ng \\ \lambda &\leq \mu_{\tilde{H}_j}(X), & j &= 1, \ldots, q \\ h_j(X) &= b_j, & j &= q+1, \ldots, nh \end{aligned} \tag{16}$$

Problem P3 is a generalization of problem P2 where p inequality constraints out of a total of ng, and q equality constraints out of a total of nh constraints are fuzzy. This problem can be solved using standard single objective nonlinear programming techniques. A combination of cubic extended interior penalty, BFGS variable metric, and golden section method are employed to solve this problem.[13]

Problem P3 is more general than problem P2 because of the presence of fuzzy constraints. Whereas problem P2 has only fuzzy objectives and crisp constraints, problem P3 has fuzzy objective functions and constraints that are partly crisp and partly fuzzy. Thus, the concept of Pareto-optimality used for crisp optimization problems needs to be revised to introduce the concept of a fuzzy Pareto-optimal solution. This is done because the solutions differ not only with respect to associated values for objective functions but also with respect to their degree of feasibility.[21] Thus the definition of Pareto-optimality is extended as follows.

Definition 4: Let f_i: $R^n \rightarrow R^1$, $i = 1, \ldots, k$ be the objective functions and $\mu_{\tilde{G}_i}$: $R^n \rightarrow [0,1]$, $i = 1, \ldots, p$, and $\mu_{\tilde{H}_i}$: $R^n \rightarrow [0,1]$, $i = 1, \ldots, q$ be the membership functions of fuzzy constraints. A solution $X^* \in S$ is said to be fuzzy Pareto-optimal, if and only if there exists no $\bar{X} \in S$, such that

$$f_i(\bar{X}) \leq f_i(X^*) \qquad \text{for all } i \in [1, \ldots, k]$$

and

$$\mu_{\tilde{G}_i}(\bar{X}) \geq \mu_{\tilde{G}_i}(X^*) \qquad \text{for all } i \in [1, \ldots, p]$$

and

$$\mu_{\tilde{H}_i}(\bar{X}) \geq \mu_{\tilde{H}_i}(X^*) \qquad \text{for all } i \in [1, \ldots, q]$$

and

$$f_j(\bar{X}) < f_j(X^*) \qquad \text{for at least one } j \in [1, \ldots, k]$$

or

$$\mu_{\tilde{G}_j}(\bar{X}) > \mu_{\tilde{G}_j}(X^*) \qquad \text{for at least one } j \in [1, \ldots, p]$$

or

$$\mu_{\tilde{H}_j}(\bar{X}) > \mu_{\tilde{H}_j}(X^*) \qquad \text{for at least one } j \in [1, \ldots, q] \tag{17}$$

It can be shown that if there exists a unique optimal solution to problem P3, then this solution is fuzzy Pareto-optimal.

III. Membership Functions

One of the major assumptions while solving fuzzy mathematical programming problems involves the use of linear membership functions for all the fuzzy sets involved in a decision-making process. A linear approximation is most commonly used because of its simplicity and expediency, and is defined by fixing two points; the upper and lower levels of acceptability.[3,15,20] If fuzzy set theory is considered to be a purely formal theory, then such an assumption is acceptable, even though some kind of formal justification of this assumption would be desirable. If, however, fuzzy set theory is used to model real world decision-making processes, some kind of empirical justification for this assumption becomes necessary. In view of this, several other (nonlinear) shapes for membership functions, such as concave or convex shaped membership functions, are analyzed to determine their impact on the overall design process. The marginal rate of increase (or decrease) of membership values as a function of design parameter values is not constant for these nonlinear membership functions, as is the case with linear membership functions. These nonlinear shapes offer potential benefits in terms of realism and are chosen consistent with varying perceptions of the decision maker (designer).

Several different shapes for the (monotonically decreasing) membership functions corresponding to the fuzzy objective functions are presented next, and later examined to determine their impact on the overall design process. These shapes correspond to what we define as positive (convex), negative (concave), or zero (linear) values of the coefficient of membership satiation, $m(X)$, which is defined as follows:

$$m(X) = \mu''(X) \tag{18}$$

where $\mu''(X)$ is the second derivative of the membership function. This definition is analogous to the Arrow-Pratt measure of risk aversion and the Dyer-Sarin measure of value satiation used in decision analysis for characterizing utility and measurable value functions, respectively.[4,10,11] It may be noted that this definition of $m(X)$ does not include $\mu'(X)$ because a linear transformation of membership functions is not possible, which is also the case with utility or value functions. A positive value of $m(X)$ corresponds to increasing marginal membership values at a given value of X (convex functions). Similarly, a negative value of $m(X)$ corresponds to a decreasing marginal membership values (concave functions), and $m(X)=0$ is equivalent to constant marginal membership values (linear functions). Second-order effects, which determine whether $m(X)$ is increasing, constant, or decreasing over the parameter range of interest, while retaining its sign, are also considered. The sine and exponential ($k>0$) functions model increasing and decreasing values of $m(X)$ over the range of definition $[m(X)>0]$. The logarithmic, quadratic, and exponential ($k<0$) functions are used to model increasing, constant, and decreasing values of $m(X)$ when the membership satiation coefficient is negative. Whereas the satiation coefficient retains its sign for these five functions, the sign of $m(X)$ changes over the range of definition for a hyperbolic function. The membership function of a fuzzy goal can also be viewed as a kind of utility function representing the degree of satisfaction or acceptance. Some of the nonlinear shapes that we have considered are shown in Fig. 2, and are discussed in detail next.

In the following five subsections dealing with different membership functions, z corresponds to a particular value of the fuzzy objective function ($\tilde{Z}$), and $z_{\min}$ and $z_{\max}$ are the fuzzy lower and upper bounds on the fuzzy objective function.

Exponential Function

An exponential membership function is defined as

$$\mu_{\tilde{Z}} = \begin{cases} 1 & \text{if } z \le z_{\min} \\ 0 & \text{if } z \ge z_{\max} \\ \dfrac{e^{-k\delta} - e^{-k}}{1 - e^{-k}} & \text{otherwise} \end{cases} \tag{19}$$

where

$$\delta = \frac{z - z_{\min}}{z_{\max} - z_{\min}} \tag{20}$$

and k is a parameter prescribed by the decision maker. When $k>0$, $\mu_{\tilde{Z}}$ is convex and consequently models an increasing marginal rate of membership values. While $m(X)$ is positive, its value decreases over the entire range of interest. A negative value of $m(X)$ can also be modeled using the preceding function for the case when $k<0$. Here again, the magnitude of $m(X)$ is decreasing over the range of definition.

Hyperbolic Function

The hyperbolic function is convex over a part of the objective function values, and is concave over the remaining part. The rationale for such a shape in our problem context is as follows. When worse off with respect to a goal, the decision maker tends to have a higher marginal rate of satisfaction with respect to that goal. A convex shape captures that behavior in the membership function. On the other hand, when one is better off with respect to a goal, one tends to have a smaller marginal rate of satisfaction. Such behavior is modeled using the concave portion of the membership function. The complete function is as follows:

$$\mu_{\tilde{Z}}(X) = 0.5 - 0.5 \tanh\left[(z - z_{\text{avg}})\delta\right] \tag{21}$$

Table 2 Results obtained with six objectives considered simultaneously

Attribute	Linear	Hyperb.	Sine	Expon. ($k=1.0$)	Quad.	Log	Expon. ($k=-1.0$)
$\epsilon \times 10^2$	3.132	4.121	3.408	3.11	2.847	2.899	3.462
F_{12}	0.1687	0.2634	0.1754	0.1422	0.1882	0.2094	0.1664
F_{14}	0.1949	0.2982	0.2191	0.1735	0.2044	0.2071	0.192
T_s	0.0777	0.0759	0.0941	0.0727	0.0801	0.0737	0.0706
SF	0.2106	0.2865	0.272	0.1986	0.1888	0.1806	0.1954
SM	0.2781	0.4662	0.2649	0.2126	0.2341	0.3029	0.2142
λ	0.989	0.9968	0.9769	0.9865	0.9989	0.9964	0.9939

Table 3 Comparison of single vs multiple objective optimization results

Attribute (1)	Structural error[a] (2)	Six objectives[b] (3)	Impr. factor[c] (4)	Kakatsios Tricamo[d] (5)	Impr. factor[e] (6)
$\epsilon \times 10^2$	2.63	3.132	0.84	8.1	2.59
F_{12}	15.38	0.1687	91.17	9.24	54.77
F_{14}	7.524	0.1949	38.61	3.61	18.52
T_s	3.654	0.0777	47.03	4.65	59.85
SF	19.34	0.2106	91.83	9.36	44.44
SM	23.61	0.2781	84.89	12.93	46.49

[a] Only structural error (f_1) is minimized.
[b] All six objectives considered simultaneously (linear memb. fns).
[c] Col.(4) = col.(2)/col.(3).
[d] Results from Table 3, Run 5 in Ref. 9.
[e] Col.(6) = col.(5)/col.(3).

where

$$\delta = \frac{6}{z_{max} - z_{min}} \tag{22}$$

This function has a membership value of 0.5 when $z = z_{avg} = 0.5*(z_{min} + z_{max})$, and is symmetric with respect to the point z_{avg}. The DM's $m(X)$ is positive and decreasing from $[z_{min}, z_{avg}]$, and is negative and increasing from $[z_{avg}, z_{max}]$, with z_{avg} being the point of inflection.

Quadratic Function

A quadratic function is used to model a negative, but constant value of $m(X)$ on part of the decision maker. The function is expressed as

$$az^2 + bz + c = \mu_{\tilde{Z}} \tag{23}$$

Assuming that

$$\mu_{\tilde{Z}} = \begin{cases} 1 & \text{if } z \le z_{min} \\ 0 & \text{if } z \ge z_{max} \\ 0.5 & \text{if } z = z_{avg} \end{cases} \tag{24}$$

the values of a, b, and c can be determined by solving these equations:

$$az_{min}^2 + bz_{min} + c = 1.0 \tag{25}$$

$$az_{max}^2 + bz_{max} + c = 0.0 \tag{26}$$

$$az_{avg}^2 + bz_{avg} + c = 0.5 \tag{27}$$

If z_{avg} is taken to be $0.5*(z_{min} + z_{max})$, the quadratic form given by Eq. (23) degenerates to a linear form as a becomes equal to zero.

Logarithmic Function

A logarithmic function is also used to model decreasing marginal rates of membership values. The function is given as

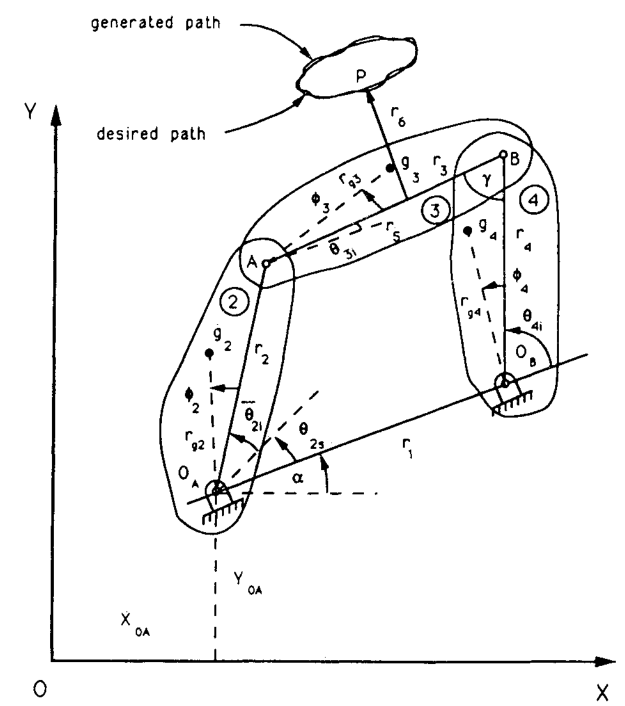

Fig. 3 Path generating planar four-bar mechanism.

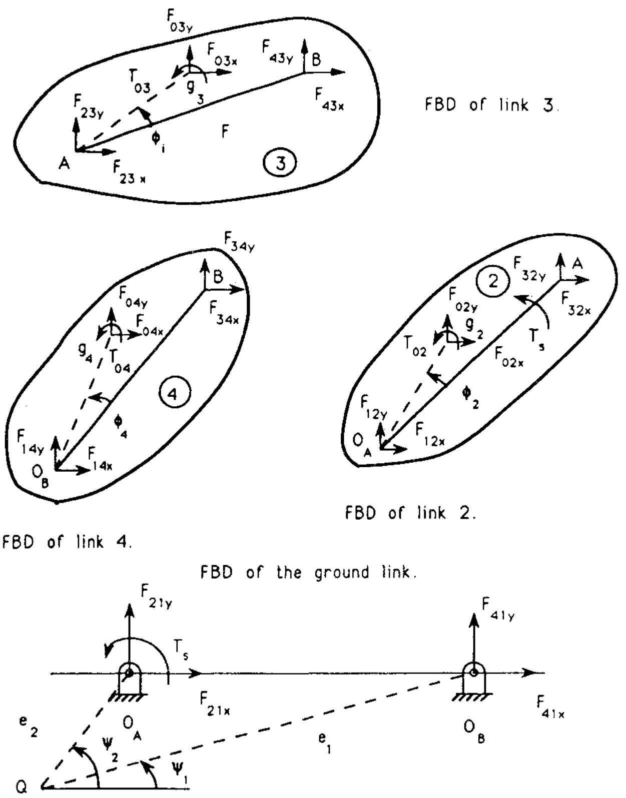

Fig. 4 Free body diagrams used in the dynamic analysis.

follows:

$$\mu_{\tilde{Z}} = \begin{cases} 1 & \text{if } z \le z_{min} \\ 0 & \text{if } z \ge z_{max} \\ a + \log(c - z) & \text{otherwise} \end{cases} \tag{28}$$

This concave function is characterized by a negative value of the membership satiation coefficient over the entire range of definition. However, the value of $m(X)$ is increasing over the parameter range of interest.

Sine Function

A sine function is used to model positive and increasing values of $m(X)$ on the part of the decision maker. This function is expressed as

$$\mu_{\tilde{Z}} = \begin{cases} 1 & \text{if } z \le z_{min} \\ 0 & \text{if } z \ge z_{max} \\ 1 - \sin \frac{\pi}{2} \delta & \text{otherwise} \end{cases} \tag{29}$$

where δ is given by Eq. (20).

IV. Design Examples

The effectiveness of fuzzy mathematical programming techniques presented in Sec. II, and the influence of nonlinear membership functions on the overall design process is demonstrated via an application to two multiobjective design problems. The first problem deals with the design of planar high-speed mechanisms in the presence of six objective functions. The second example addresses the optimum design of space structures where three objective functions are considered simultaneously.

Design Example I

A new integrated approach to the design of high-speed planar mechanisms is investigated. The resulting nonlinear

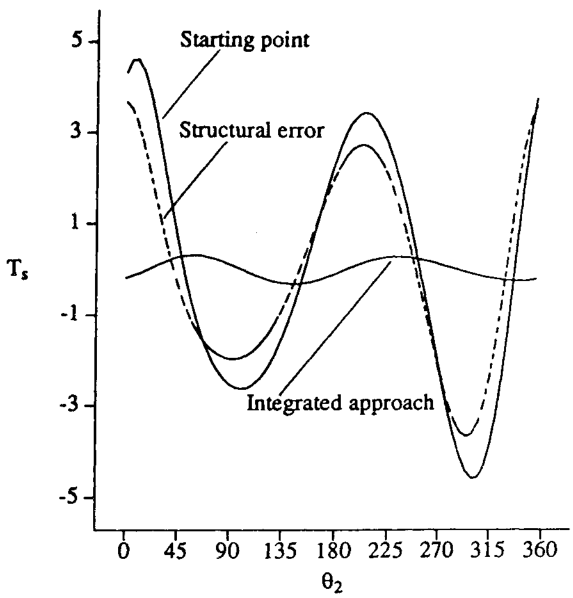

Fig. 5 Driving torque as a function of input link orientation.

programming formulation combines both the kinematic and dynamic synthesis aspects of mechanism design. The multiobjective optimization techniques facilitate the design of a linkage to meet several kinematic and dynamic criteria. A total of six (one kinematic and five dynamic) criteria are considered. The kinematic criterion deals with the minimization of the difference between the desired motion and generated motion. The dynamic criteria consist of the minimization of input driving torque, ground bearing forces, and the shaking forces and shaking moments transmitted to the ground link over a cycle. The mathematical formulation of the design problem is given next.

A four-bar mechanism is to be synthesized to generate a given path with coordinated rotation of the input link. Using Fig. 3, the coordinates of the path traced by the coupler point P are given as

$$X_{gi} = X_{O_A} + r_2 \cos(\theta_{2i} + \alpha) + r_5 \cos(\theta_{3i} + \alpha) - r_6 \sin(\theta_{3i} + \alpha) \tag{30}$$

$$Y_{gi} = Y_{O_A} + r_2 \sin(\theta_{2i} + \alpha) + r_5 \sin(\theta_{3i} + \alpha) + r_6 \cos(\theta_{3i} + \alpha) \tag{31}$$

where

$$\theta_{2i} = \theta_{2s} + \bar{\theta}_{2i} \tag{32}$$

Here X_{O_A} and Y_{O_A} denote the coordinates of the ground pivot O_A, α is the angular orientation of the ground link, r_i $(i = 1, \ldots, 6)$ are the link lengths, θ_{2s} is the starting position of the input link, and θ_{2i} and θ_{3i} are the angular orientations of links 2 and 3 at the ith design position, respectively. Let the desired (prescribed) values of the path coordinates be given as (X_{di}, Y_{di}). The first objective (f_1) considered is a minimization of the structural error over the entire range of motion, and is defined as

$$f_1 = \sum_{i=1}^{N} \epsilon_i^2 + \sum_{i=1}^{N} \left[(X_{di} - X_{gi})^2 + (Y_{di} - Y_{gi})^2 \right] \tag{33}$$

where N denotes the number of design points into which the path is divided. A minimization of f_1 can be achieved by varying the link lengths $r_1 - r_6$, and the ground pivot parameters X_{O_A}, Y_{O_A}, and α.

The following behavior constraints are imposed on the design problem:

1) The mechanism must satisfy the loop closure equation at each design position. This is achieved by using an equality constraint of the form

$$2r_2r_4 \cos(\theta_{2i} - \theta_{4i}) - 2r_1r_4 \cos\theta_{4i} + 2r_1r_2 \cos\theta_{2i} + r_3^2 = r_1^2 + r_2^2 + r_4^2 \tag{34}$$

at each design position.

2) The structural error at each design position is constrained to be less than a specified quantity Δ, i.e.,

$$\epsilon_i \le \Delta \qquad i = 1, \ldots, N \tag{35}$$

3) A further design restriction that assures that the input link be a crank can be stated as

$$r_1 + r_2 < r_3 + r_4 \tag{36}$$

$$[r_3 - r_4]^2 < [r_1 - r_2]^2 \tag{37}$$

4) The value of transmission angle γ over the entire cycle is constrained as

$$\pi/6 \le \gamma \le (5\pi)/6 \tag{38}$$

5) Last, to ensure that each solution is free of a branch defect, an inequality constraint relating the sign of γ_0 and γ_j is imposed at each design position.[2]

In the present work, a value of $N = 10$ is used and the coordinates of the prescribed path are as follows:

$$X_{di} = 0.4 - \sin 2\pi(t_i - 0.34) \tag{39}$$

$$Y_{di} = 2.0 - 0.9 \sin 2\pi(t_i - 0.5) \tag{40}$$

where

$$t_i = (i - 1)/N \tag{41}$$

The coordinated input link orientations are determined using

$$\bar{\theta}_{2i} = 2\pi t_i \tag{42}$$

The dynamic analysis procedure used in this study is valid for the general four-bar linkage shown in Fig. 3. The rigid links are assumed to have a general shape, and the revolute joints are considered to be frictionless. Each of the links has a

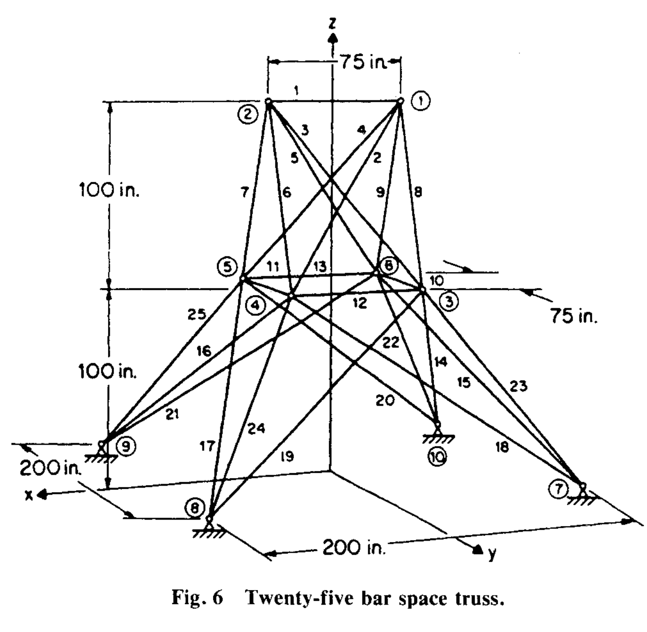

Fig. 6 Twenty-five bar space truss.

Table 4 Loading conditions for the 25-bar truss

	Joint			
	1	2	3	6
	Loading condition 1			
F_x	0	0	0	0
F_y	20,000	−20,000	0	0
F_z	−5,000	−5,000	0	0
	Loading condition 2			
F_x	1000	0	500	500
F_y	10,000	10,000	0	0
F_z	−5,000	−5,000	0	0

Table 5 Results for single-objective optimizations

Attribute	Starting point	Min. weight	Min. deflec.	Max. freq.
Weight	451.308	233.4154	1653.6036	685.8162
Deflection	1.1045	1.92973	0.30833	1.3308
Frequency	73.135	73.1071	68.8651	110.146

length r_i, $i = 1, \ldots, 4$, and each of the moving links has a mass m_i and a moment of inertia I_i, $i = 2, 3, 4$ with respect to the center of mass, which is defined by r_{gi} and ϕ_i (Fig. 3). Using the free body diagrams for the links given in Fig. 4, the equations of motion for each of the three moving links can be written. This results in a system of nine simultaneous equations with nine unknowns. Since all inertia forces (F_{oix}, F_{oiy}) and inertia couples (T_{oi}) in these equations are known, one can solve this system of simultaneous equations for the x and y components of the four bearing reactions ($F_{12}, F_{23}, F_{34}, F_{14}$) and the input driving torque (T_s).

The shaking force (SF) defined as the resultant force on the ground link is now given as

$$\mathrm{SF} = F_{21} + F_{41} \tag{43}$$

where SF_x and SF_y are the x and y components of the shaking force. Alternately, by using the nine equilibrium equations, one gets

$$\mathrm{SF}_x = F_{02x} + F_{03x} + F_{04x} \tag{44}$$

$$\mathrm{SF}_y = F_{02y} + F_{03y} + F_{04y} \tag{45}$$

The shaking moment (SM) about an arbitrary point Q on the ground link is given as

$$\mathrm{SM} = -T_s - F_{41x}e_1 \sin\psi_1 + F_{41y}e_1 \cos\psi_1 - F_{21x}e_2 \sin\psi_2 + F_{21y}e_2 \cos\psi_2 \tag{46}$$

When Q is the midpoint of link O_AO_B (Fig. 4),

$$e_1 = e_2 = r_1/2$$

$$\psi_1 = 0, \qquad \psi_2 = 180$$

and the expression for the SM reduces to

$$\mathrm{SM} = (r_1/2)(F_{12y} - F_{14y}) - T_s \tag{47}$$

The dynamic analysis is performed at every five-degree rotation of the input link. This results in a total of 72 evaluations during each cycle of crank rotation. The ultimate objective is to design a mechanism that requires the minimum driving torque, and that transmits minimum forces and moments to the ground. Thus, two objective functions are selected as $f_2 = \max F_{12}$ and $f_3 = \max F_{14}$, where $\max F_{12}$ and $\max F_{14}$ are the maximum values of ground bearing forces realized during one input crank revolution. The next dynamic criterion, f_4, is taken as the max T_s, or the maximum value of input driving torque required over a cycle. Finally the last two objective functions, f_5 and f_6, are chosen as the maximum values of SF and SM over a cycle, respectively. Thus, the optimization problem has a total of six objective functions.

The six objectives included in the problem formulation are optimized by varying 18 different mechanism parameters. They consist of the six link lengths (r_1–r_6), three ground pivot O_A parameters, namely X_{O_A}, Y_{O_A}, and α (Fig. 3). In addition, one counterweight is attached to each of the three moving links (2, 3, 4) to minimize the SFs and SMs transmitted to the ground link. The counterweight radii, thicknesses, and orientations yield an additional set of nine design variables. The design problem is subject to a total of $3 \times N + 4$ constraints corresponding to the N design points into which the desired path is divided. For this design example, only the objective functions are assumed to be fuzzy.

The single-objective optimization problems are solved first, and the optimum values of the objective functions are given in Table 1. Using the results of the single-objective optimizations, the procedure outlined under the previous subsection, "Computational Procedure," is followed, and the membership functions for the six fuzzy goals are constructed. This results in a problem similar to problem P2, with a total of 19 design variables and 40 constraints. The results obtained by solving this fuzzy optimization problem are summarized in Table 2. Table 2 also presents the results obtained when nonlinear membership functions are employed for the six fuzzy objective functions.

It can be observed from Table 3 and Fig. 5 that, when all six objectives (f_1–f_6) are considered simultaneously, the dynamic characteristics of the resulting mechanism exhibit a remarkable improvement over the linkage for which only the structural error is optimized. The improvement factors for the dynamic performance measures range anywhere from 38.6 to 91.8. The structural error has increased by a small amount ($1/0.84 = 1.19$) compared to the case when only structural error is minimized. Thus, for this planar mechanism design problem, substantially improved kinematic and dynamic characteristics can be obtained when all the kinematic and dynamic attributes are considered simultaneously.

The results obtained using the proposed multiobjective formulation also represent a significant improvement over those obtained by Kakatsios and Tricamo.[9] It can be seen from Table 3 that the improvement factors range anywhere from 2.59 (for structural error) to 59.85 (for input driving torque) when comparing the results obtained by Kakatsios and Tricamo to the results obtained using the multiobjective formulation presented herein. Thus, prior formulations that have used only a single kinematic or dynamic attribute as an objective function, and treated the remaining attributes using inequality constraints, can lead to inferior solutions. This results from the fact that the conflicting and competing nature of several kinematic and dynamic criteria seldom permits one to select a single attribute as an objective function.

When nonlinear shapes are employed for the membership functions corresponding to the fuzzy objective functions, the kinematic and dynamic performance measures of the resulting mechanism exhibit similar trends for various values of the membership satiation coefficient (see Tables 2 and 3). The results obtained using quadratic, exponential ($k < 0$), and logarithmic membership functions are similar, as these three functions model a negative value of $m(X)$. However, the results obtained using these three functions are fairly different from those obtained with membership functions that model constant or increasing marginal membership satiation values. The difference in the results obtained using quadratic, exponential ($k < 0$), and logarithmic functions is attributed to second-order effects. Even though these three functions have the same sign for the membership satiation coefficient [$m(X) < 0$],

Table 6 Results with all three objectives considered simultaneously

Attribute	Linear	Hyperb.	Sine	Expon. ($k = 1.0$)	Quad.	Log	Expon. ($k = -1.0$)
Weight	533.909	622.518	535.102	526.591	555.653	539.401	543.34
Deflection	0.9202	0.855	0.9257	0.9292	0.8953	0.915	0.9175
Frequency	94.526	96.2	94.57	94.35	95.145	94.626	94.662
λ	0.6226	0.876	0.4365	0.4977	0.7901	0.7296	0.7356

the value of $m(X)$ is constant, decreasing and increasing for these functions. However, the second-order effects are not as dominant as the first-order effects, which govern the sign of the membership satiation coefficient. Hence, it is important to accurately assess the nature of the membership functions (e.g., concave, convex, or linear), i.e., the sign of the membership satiation coefficient influences the optimum results significantly. The reference lottery approach widely used in multiattribute utility theory can be used to accomplish this objective.[4,11]

Design Example 2

Consider the 25-bar truss shown in Fig. 6. This truss is required to support two loading conditions given in Table 4 and is to be designed with constraints on member stresses and Euler buckling. The allowable stress for all members is specified as S, in both tension and compression. The Young's modulus and the material density are taken as $E = 1.0 \times 10^7$ psi and $\rho = 0.1$ lb/in.^{3}, respectively. The members are assumed to be tubular, with a nominal diameter to a thickness ratio of 100, so that the buckling stress in the ith member becomes

$$p_i = \frac{-100.01\pi E A_i}{8 l_i^2}, \qquad i = 1,\ldots,25 \tag{48}$$

where A_i and l_i denote the cross-sectional area and length, respectively, of member i. The member areas are linked in the following groups: A_1, $A_2 = A_3 = A_4 = A_5$, $A_6 = A_7 = A_8 = A_9$, $A_{10} = A_{11}$, $A_{12} = A_{13}$, $A_{14} = A_{15} = A_{16} = A_{17}$, $A_{18} = A_{19} = A_{20} = A_{21}$, and $A_{22} = A_{23} = A_{24} = A_{25}$. Thus, there are a total of eight independent design variables in the problem specification. Three objective functions, namely, the minimization of weight, the minimization of deflection of nodes 1 and 2, and the maximization of fundamental natural frequency of vibration of truss, are considered. The objective functions are expressed as

$$f_1(X) = \sum_{i=1}^{25} \rho A_i l_i \tag{49}$$

$$f_2(X) = (\delta_{1x}^2 + \delta_{1y}^2 + \delta_{1z}^2)^{1/2} + (\delta_{2x}^2 + \delta_{2y}^2 + \delta_{2z}^2)^{1/2} \tag{50}$$

$$f_3(X) = -\omega_1 \tag{51}$$

where δ_{ix}, δ_{iy}, and δ_{iz} denote the x, y, and z components of deflection of node i ($i = 1, 2$), and ω_1 indicates the fundamental natural frequency of vibration. The constraints in the crisp problem specification are stated as follows:

$$|\sigma_{ij}| \le S, \qquad i = 1,\ldots,25, \qquad j = 1,2 \tag{52}$$

$$\sigma_{ij} \ge p_i, \qquad i = 1,\ldots,25, \qquad j = 1,2 \tag{53}$$

$$x_i^l \le x_i \le x_i^u, \qquad i = 1,\ldots,8 \tag{54}$$

where σ_{ij} denotes the stress in member i under load condition j, S is the allowable stress, taken as 40,000 psi, p_i is the buckling stress given by Eq. (48), x_i^l and x_i^u are the lower and upper bounds on x_i, assumed to be 0.1 and 5.0 in.^{2}, respectively, for $i = 1,\ldots,8$.

The single-objective optimization problems are solved first and the results are given in Table 5. The best and worst values of each of the objective functions can be identified from these results. These values aid the construction of membership functions of fuzzy objective functions. For this example, the design constraints given by Eqs. (52–54) are considered to be partly crisp and partly fuzzy. A constraint leeway of 4000 psi and 0.01 in.^{2} is permitted for constraints limiting the magnitude of σ_{ij} and the lower bound on x_i. Design constraints limiting the buckling stress and maximum value of x_i are assumed to be crisp. The step-by-step procedure outlined under the "Computational Procedure" subsection is adopted to construct membership functions corresponding to fuzzy constraints. This results in an optimization problem similar to problem P3, with a total of nine design variables and 119 constraints. While solving this problem, it is assumed that objective function f_1 is twice as important as objectives f_2 and f_3. The relative importance of objective functions is modeled using the concept of linguistic hedges.[18]

Design example 2 is more general compared to example 1 because of the presence of fuzzy constraints. Whereas example 1 has only fuzzy objective functions and crisp constraints, example 2 consists of fuzzy objective functions and constraints that are partly crisp and partly fuzzy. Thus the concept of a fuzzy Pareto-optimal solution is needed for this optimization problem. The results obtained by solving the resulting fuzzy optimization problem are presented in Table 6 for linear as well as nonlinear membership functions. The results obtained using quadratic, exponential ($k < 0$), and logarithmic membership functions are similar, as these three functions model a negative value of $m(X)$. However, the results obtained using these three functions are fairly different from those obtained with membership functions, which model constant or increasing marginal membership satiation values. Hence, it is important to accurately assess the nature of the membership functions (e.g., concave, convex, or linear), i.e., the sign of the membership satiation coefficient influences the optimum results significantly. Second-order effects that govern whether $m(X)$ is increasing, decreasing, or constant over the parameter range of interest do not exercise a pronounced influence on the optimum solution.

V. Conclusions

Two problems dealing with the multiple objective design of mechanical and structural systems containing fuzzy information are considered. Solution methodologies for solving fuzzy mathematical programming problems using nonlinear programming techniques are presented. The concept of a Pareto-optimal and fuzzy Pareto-optimal solution is discussed, and it is shown that the resulting formulation yields Pareto-optimal solutions. A new integrated approach to synthesize planar mechanisms is presented, and is found to yield designs with superior kinematic and dynamic characteristics compared to prior formulations. Several nonlinear shapes for membership functions of the fuzzy sets are considered consistent with varying perceptions of the designer to determine their influence on the overall design process. It is seen that the optimum designs for both examples are strongly influenced by the sign of the membership satiation coefficient. Second-order effects that govern whether the satiation coefficient is increasing, decreasing, or constant over the parameter range of interest are found not to influence the results significantly.

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